Complex Analysis and Operator Theory (2023) 17:84
https://doi.org/10.1007/s11785-023-01384-2

Complex Analysis and Operator Theory

Unconditional Basis Constructed from Parameterised Szegö Kernels in Analytic $\mathbb{H}^p(D)$

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Received: 15 May 2023 / Accepted: 2 July 2023
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Abstract
Rational orthogonal systems in approximating analytic functions have attracted considerable interest. Among which adaptive Fourier decomposition, abbreviated as AFD, was recently established. An AFD is a sparse representation using a Takenaka–Malmquist (TM) system whose parameters are optimally selected according to the given signal. TM systems have been proved to be Schauder systems in the corresponding Banach spaces $\mathbb{H}^p$, $1 < p < \infty$. In the present paper, from the methodology point of view we give an alternative definition of the Hardy spaces by using the periodic Lusin area function. We extend the Botchkariev–Meyer–Wojtaszczyk Theorem to rational function systems. By using Meyer's bimodal wavelet and the Fefferman–Stein vector valued maximum operator we prove that under certain conditions the rational systems become unconditional bases in the Banach space $\mathbb{H}^p(D)$, $1 < p < \infty$.

Keywords Rational approximation · Analytic functions · Wavelets · AFD and greedy algorithm · Unconditional basis · Vector valued maximum function

Mathematics Subject Classification 46B70 · 46E35 · 42B35

Communicated by Irene Sabadini.

Co-first authors: Chitin Hon and Haibo Yang.

This article is part of the topical collection "Harmonic Analysis and Operator Theory" edited by H. Turgay Kaptanoglu, Andreas Seeger, Franz Luef and Serap Oztop.

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Published online: 26 July 2023

1 Motivations and Main Theorem

As a notion of physics frequency possesses non-negativity. While in mathematics, the characteristic that frequencies are positive corresponds to analytic functions. As a consequence, study of analytic functions has considerable significance. Orthogonal rational systems, or Takenaka–Malmquist (TM) system [19], contains the Laguerre basis and the two-parameter Kautz basis [1] as two particular cases. They are all rational orthonormal systems and natural generalizations of the half-Fourier system. Further, Pereverzyev [10] used reproducing kernel methods to deal with learning algorithms, and systematically developed reproducing kernel Hilbert spaces (RKHS). Since we often meet with Banach spaces in the learning algorithms, we need to consider reproducing Banach space (RKBS) as well.

Schauder basis was introduced in 1927. Banach and Orlicz considered unconditional basis in Banach spaces. The differences between the above two types of bases can be seen by the following definitions. See [6, 23].

Definition 1.1 (i) Let X be a Banach space. A sequence of vectors $\{e_k\}_{k=1}^{\infty}$ in X is called a Schauder basis of X, if for any element $f \in X$, there exists correspondingly a unique sequence of complex numbers $\{a_k\}_{k=1}^{\infty}$ such that

$$f = \sum_{k=1}^{\infty} a_k e_k,$$

where the convergence is in the X-norm sense, that is

$$f = \lim_{N\to\infty} S_N(f), \quad S_N(f) = \sum_{k=1}^{N} a_k e_k.$$

(ii) A sequence of vectors $\{e_k\}_{k=1}^{\infty}$ is called an unconditional basis in Banach space X, if whenever series $\sum_{k=1}^{+\infty} a_k e_k$ converges, then the series $\sum_{k=1}^{+\infty} b_k e_k$ also converges for all $\{b_k\}_{k=1}^{\infty}$ satisfying $|b_k| \le |a_k|$.

Qian and Wang developed Core AFD in 2011 [15]. Their group further generalized the theory to include a number of variations [11–14, 17, 24]. The contexts that they study range from the unit disk to lately the irreducible bounded symmetric domain. They use matching pursuit methodology to the kernel functions and their derivatives. Then Gram-Schmidt orthogonalization processes are used to construct orthogonal systems. They worked on attainability of adaptively optimal parameters of Takenaka–Malmquist (TM) systems. A general term of a TM system is the product of a Szegö kernel and a finite Blaschke product in which the involved parameters can have multiplicity. TM system was introduced in 1925. Qian-Chen-Tan [14] and Wang-Qian [21] proved that in $\mathbb{H}^p$, $1 < p < \infty$, TM systems are Schauder systems of the closures of their spans.

To our best knowledge, for $p \neq 2$, there have been no results connecting TM system with unconditional basis in $\mathbb{H}^p (1 < p < \infty)$. In data processing, unconditional bases have remarkable convenience and advantages. In this paper, we show that under suitable arrangement of locations of the parameters of a TM system and replacement of the Blaschke products by differences of certain rational functions, unconditional bases can be constructed in $\mathbb{H}^p(D)(1 < p < \infty)$. As an application, our rational systems can extend Pereverzyev's reproducing kernel Hilbert spaces (RKHS) to reproducing kernel Banach spaces (RKBS).

Botchkariev [2] and Wojtaszczyk [22] applied Franklin system to analyze $\mathbb{H}^p(D)$. Afterwards, Meyer [9] used bimodal wavelets to restudy $\mathbb{H}^p(D)$ where the basis is unconditional. But Meyer's bimodal wavelets have a very complicated structure. Rational functions, which come from the Cauchy formula, have a simple and intuitive expression. In this paper, we hope to find a compromise method combining TM system and wavelets. Our main idea is to use the discrete form of Lusin's periodized area integral. We prove that there exists quasi-orthogonality between our basis and Meyer's wavelet basis. The constructed rational basis is eventually an unconditional basis for $\mathbb{H}^p(D)$.

Adopting the idea used in [7, 16, 25], an index $m \geq 2$ may be expressed by two indices j and k satisfying $m = 2^{j-1} + 1 + k$, where $j \geq 1, 0 \leq k < 2^{j-1}$. When m, j, k satisfy the above relation we write $m \sim (j, k)$. The index j represents the approximate range of frequency and the index k represents the approximate location. These two indices coincide with the characteristics of an unconditional basis.

Let $\chi_I(x)$ be the characteristic function of the set I. When $I = [0, 1]$, we abbreviate $\chi_I(x)$ as $\chi(x)$. Hence $\chi(2^j x - k)$ is the characteristic function of the interval $[2^{-j}k, 2^{-j}(k+1)]$ and $\chi(2^j x - x_m^f)$ is the characteristic function of the interval $I_{m,f} = [2^{-j}x_m^f, 2^{-j}(1 + x_m^f)]$. In Sect. 4, we will construct $\{H_m\}_{m\geq 0}$, which is composed by certain analytic rational functions in the upper half disc. In Sect. 6.1, we will construct $\{\tilde{H}_m\}_{m\geq 0}$ which is the dual basis of $\{H_m\}_{m\geq 0}$. Our main theorem is as follows:

Theorem 1.2 *(i) For $1 < p < \infty$, there exist two basis $\{H_m\}_{m\geq 0}$ and $\{\tilde{H}_m\}_{m\geq 0}$ where $\{H_m\}_{m\geq 0}$ is composed by certain analytic rational functions in the upper half disc and $\{\tilde{H}_m\}_{m\geq 0}$ is its dual basis, such that, for all analytic function $f \in \mathbb{H}^p(D)$ on the disc D, the following equality holds in $\mathbb{H}^p(D)$:*

$$f = \sum_m \langle f, \tilde{H}_m \rangle H_m.$$

Further, $\{H_m\}_{m\geq 0}$ is an unconditional basis and **the absolute value** *of the coefficients can reflect the norm property*

$$\|f\|_{\mathbb{H}^p} \sim |f_0| + |f_1| + \left\| \left(\sum_{2\leq m\sim(j,k)} |\langle f, \tilde{H}_m \rangle|^2 2^j \chi_{[2^{-j}k, 2^{-j}(k+1)]} \right)^{\frac{1}{2}} \right\|_{L^p}, \tag{1.1}$$

where $f_0 = \langle f, \tilde{H}_0 \rangle$ and $f_1 = \langle f, \tilde{H}_1 \rangle$. For each m in equation (1.1), there holds

$$|\langle f, \tilde{H}_m \rangle| \leq C m^{\frac{1}{p} - \frac{1}{2}} \|f\|_{\mathbb{H}^p}. \tag{1.2}$$

(ii) *The unconditional basis* $\{H_m\}_{m\geq 0}$ *allows us to make a maximal selection for* $1+2^{j-1} \leq m \leq 2^j$ *at each* $j \geq 1$. *Given* $f \in \mathbb{H}^p(D)$. *In fact, for* $0 \leq m \leq 2$, *we take* $H_m^f = H_m$; *and for* $m \geq 3$, **take** x_m^f **to be a maximal choice** *referred to the analysis in Sect. 6.1. We get a rearrangement* $\{H_m^f\}_{m\geq 3}$ *of* $\{H_m\}_{m\geq 3}$ *located at* $I_{m,f}$ *which provide a fast converging expansion in the sense of* $\mathbb{H}^p(D)$

$$f = \sum_{m\geq 0} \langle f, \tilde{H}_m^f \rangle H_m^f. \tag{1.3}$$

In order to **highlight typicality of the method**, this paper considers only the unit disc D case with its Szegö kernels, and for the special Banach spaces $H^p(D)$. The methods and results can, in fact, be generalized to more general function spaces (like Besov spaces, Hausdorff spaces and Q spaces etc). The unit disk D may be generalized to bounded symmetric fields [24], as well as to several real variables (under Clifford algebra setting) and several complex variables and even matrix-valued functions.

Remark 1.3 For Banch spaces such as analytic $H^p(D)$ spaces, we establish the unconditional basis in (i). Skezypczak [18] used spline functions to construct unconditional basis on real $\mathbb{H}^p(D)$. Calderon-Zygmund operators have relation to unconditional basis. See also Garca-Cuerva and Kazarian [6] for Calderon-Zygmund operators. We have established a partial greedy algorithm in (ii) for analytic $H^p(D)$ spaces which are Banach spaces. For non-analytic Hilbert space, Mallat et al. [4, 8] used time-frequency method to make adaptive greedy approximation to signals.

Remark 1.4 (i) In this paper, as a crucial technical method the unconditional basis is constructed and proved via certain quasi-orthogonality. We do not know now whether the systems generated by the Blaschke products in the work [11–13] of Qian et al. have quasi-orthogonality with our basis. Hence we cannot prove so far whether there is an unconditional basis for $H^p(D)(p \neq 2)$ formed by Blaschke products.

(ii) According to the maximum choice principle, $|\langle f, \tilde{H}_m^f \rangle|$ is **bigger** than $|\langle f, \tilde{H}_m \rangle|$. Therefore, the convergence rate of algorithm (1.3) is faster than which in the equation (1.2).

The rest of this article is structured as follows:

In Sect. 2, we present some preliminaries on $\mathbb{H}^p(D)$ and the Cauchy formula. In Sect. 3, we present preliminaries on discretization via wavelets. In Sect. 4, we establish BMW Theorem for rational function system. We introduce first some rational function system, and then establish the pseudo-orthogonality among those rational functions. Then we establish also the pseudo-orthogonality between the rational system and Meyer bimodal wavelets. Further, we apply Fefferman-Stein vector value maximum operator theorem to establish the relation between Hardy spaces defined by rational system and those defined by wavelets. In Sect. 5, we prove completeness of our rational function system. In Sect. 6, firstly, we present an AFD analogue and then prove Main Theorem 1.2.

2 Preliminaries on $\mathbb{H}^p(D)$ and Cauchy Formula

In the half-plane case, $\mathbb{H}^p(\mathbb{R}^2_+)$ is the space formed by the analytic function $F(z) = F(x+iy)$ satisfying the condition

$$\sup_{y>0}\int |F(x+iy)|^p dx < \infty.$$

In the half plane, the Cauchy formula is written as:

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(y)}{y-z}dy,\ \mathrm{Im}z > 0.$$

The real and imaginary parts of the Cauchy kernel correspond to the following kernel functions: $P_t(x) = \frac{t}{\pi(t^2+x^2)}$ and $Q_t(x) = \frac{x}{\pi(x^2+t^2)}$. Let I be the unit operator. Let H be the Hilbert transform, it can be shown

$$\hat{H}f(\xi) = -i\,\mathrm{sgn}\xi\,\hat{f}(\xi).$$

Then

$$F(x) = \frac{1}{2}f(x) + i\frac{1}{2}Hf(x)$$

is an analytic function. $P = \frac{1}{2}(I + iH)$ is the orthogonal projection from $L^2(\mathbb{R})$ to $\mathbb{H}^2(\mathbb{R})$. The derivative of the Cauchy kernel $\frac{1}{y-z}$ can be approximated by an analytical function formed by a special sequence of points in the region. Since the derivative reflects the vanishing moment property of the boundary function, the distance from the point to the boundary reflects that the analytical function is concentrated near the corresponding frequency. The $\mathbb{H}^p(\mathbb{R}^2_+)$ norm can be described by using Lusin area function by appropriately selecting the distances between sampling points and the boundary and between sampling points themselves. See Meyer's Wavelets and Operators in Volume 1, Chapter 1, Section 5. The reason why the analytic function space can be characterized is that the matrix corresponding to the inner product between the corresponding function and the wavelets satisfies the estimation of Eq. (3.1) of Proposition 1 in Section 3, Chapter 8, Volume 2 of Meyer's book [9].

Now we turn to analytic functions on the unit disk that have power series expansions $F(z) = \sum_0^\infty C_k z^k,\ |z| < 1$. Their boundary values may be written $f(t) = F(e^{2\pi it})$.

Definition 2.1 We call $f(x) \in \mathbb{H}^2[0,1]$, if there exists $\{C_K\}_{k\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}} |C_k|^2 < \infty$ and, in the L^2 convergence sense,

$$f(x) = \sum_{k\in\mathbb{N}} C_k e^{2\pi ikx}.$$

The space of analytic functions $\mathbb{H}^p(D)$ is a Banach space composed of analytic functions defined by

$$\mathbb{H}^p(D) = \left\{ F(z) \text{ analytic}, \sup_{0 \le r < 1} \int_0^1 |F(re^{2\pi i t})|^p dt < \infty \right\}.$$

Proposition 2.2 *For $1 < p < \infty$, let $\frac{1}{p} + \frac{1}{p'} = 1$. The dual space of Hardy space $\mathbb{H}^p(D)$ is Hardy space $\mathbb{H}^{p'}(D)$.*

In the case of the unit disk, the analytic function is limited to the boundary and can be treated as a function on the interval. We use the method of periodizing the function on the real axis. For the unit disc, the above Cauchy formula corresponds to the following:

$$F(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(y)}{y - z} dy.$$

Denote

$$P_r^{[0,1]}(x) = \frac{1 - r^2}{1 - 2r \cos 2\pi x + r^2} \text{ and } Q_r^{[0,1]}(x) = \frac{2r \sin 2\pi x}{1 - 2r \cos 2\pi x + r^2}. \quad (2.1)$$

We have

$$F(re^{2\pi i x}) = \frac{1}{2} \int_0^1 [P_r^{[0,1]}(x - y) + i Q_r^{[0,1]}(x - y)] f(e^{2\pi i y}) dy, \ \forall 0 \le r < 1.$$

The Cauchy formula tells us that the characteristic of the analytic function on the region can be obtained by the characteristic of the analytic function on the boundary. In the case of the disk, although it should be similar to the case of the half-plane, there is a periodization process that induces considerable complications.

3 Preliminaries on Discretization via Wavelets

Garca-Cuerva and Kazarian [6] considered Calderon-Zygmund operators and unconditional basis of weighted Hardy spaces. Skrzypczak [18] considered some remarks on spline unconditional basis in real Hardy space $\mathbb{H}^1(D)$. In this section, we introduce some preliminaries on Meyer wavelets and analytic $\mathbb{H}^p(D)$.

3.1 Non Analytic Cases

We introduce first the necessary knowledge of the usual Meyer wavelets. See mainly Meyer [9], Qian-Yang [16], Yang [26, 27] and Yang–Cheng–Peng [28].

3.1.1 Wavelets on the Real Line

Let $\Psi(\xi)$ be an even function belonging to $C_0^\infty[-\frac{4}{3}\pi, \frac{4}{3}\pi]$ satisfying

$$\begin{cases} 0 \le \Psi(\xi) \le 1, & \forall \xi \in \mathbb{R}; \\ \Psi(\xi) = 1, & \forall |\xi| \le \frac{2}{3}\pi; \\ \Psi^2(\xi) + \Psi^2(2\pi - \xi) = 1, & \forall 0 \le \xi \le 2\pi. \end{cases}$$

Then $\omega(\xi) = \sqrt{\Psi^2(\frac{\xi}{2}) - \Psi^2(\xi)}$ is an even function belonging to $C_0^\infty[-\frac{8}{3}\pi, \frac{8}{3}\pi]$ satisfying

$$\begin{cases} 0 \le \omega(\xi) \le 1, & \forall \xi \in \mathbb{R}; \\ \omega(\xi) = 0, & \forall |\xi| \le \frac{2}{3}\pi; \\ \omega^2(\xi) + \omega^2(2\xi) = \omega^2(\xi) + \omega^2(2\pi - \xi) = 1, & \forall \frac{2}{3}\pi \le \xi \le \frac{4}{3}\pi. \end{cases}$$

Let $\hat{\psi}^0(\xi) = \Psi(\xi)$, $\hat{\psi}(\xi) = e^{-\frac{i\xi}{2}}\omega(\xi)$ and $\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \forall j, k \in \mathbb{Z}$. We have

Lemma 3.1 *(i)* $\sum_{k\in\mathbb{Z}} \psi^0(x-k) = 1$.

(ii) $\psi(x)$ *is a real value function belonging to* $\mathcal{S}(\mathbb{R})$.

(iii) $\text{Supp}\hat{\psi}(\xi) \subset \{\xi : \frac{2\pi}{3} \le |\xi| \le \frac{8\pi}{3}\}$.

(iv) $\psi(1-x) = \psi(x)$.

(v) $\{\psi^0(x-k)\}_{k\in\mathbb{Z}} \bigcup \{\psi_{j,k}(x)\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ *and* $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}}$ *are two orthogonal basis of* $L^2(\mathbb{R})$.

3.1.2 Wavelets on the Interval

For $j \in \mathbb{N}$, define

$$\begin{aligned} g_j(x) &= (2\pi)^{-1}2^{-\frac{j}{2}} \sum_{l\in\mathbb{Z}} \hat{\psi}(2l\pi 2^{-j}) e^{2l\pi i x} \\ &= (2\pi)^{-1}2^{-\frac{j}{2}} \sum_{l\in\mathbb{Z}} \omega(2l\pi 2^{-j}) e^{2l\pi i(x - 2^{-j-1})}. \end{aligned}$$

Since ω has compact support, the above sum is a finite sum for $\frac{1}{3}\cdot 2^j \le |l| \le \frac{4}{3}\cdot 2^j$. Define $g_0(x) = 1$ and for $j \in \mathbb{N}$, $0 \le k < 2^j$, and

$$g_{j,k}(x) = 2^{\frac{j}{2}} \sum_{l=-\infty}^{+\infty} \psi(2^j x + 2^j l - k) = g_j(x - k2^{-j}).$$

The function $g_{j,k}(x)$ is real-valued and symmetric about $x = 2^{-j}k + 2^{-j-1}$. Let $\Lambda = \{0\} \bigcup \{(j,k), j \in \mathbb{N}, 0 \le k < 2^j\}$.

Lemma 3.2 *(i)* $g_\lambda(x)(\lambda \in \Lambda)$ *are functions of period 1.*

(ii) $g_\lambda(x)(\lambda \in \Lambda)$ *is an orthonormal basis in* $L^2[0,1]$.

3.2 Analytic Wavelets

We present then the necessary knowledge of the Meyer bimodal wavelets. See mainly Meyer [9] and Qian-Yang [16].

3.2.1 Holomorphic Wavelets on the Real Line

We introduce first Meyer's holomorphic wavelets $\{S_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$. Denote

$$\operatorname{sgn}(\xi) = \begin{cases} -1, & \text{if } \xi < 0; \\ 1, & \text{if } \xi \geq 0. \end{cases} \quad \chi_{+}(\xi) = \begin{cases} 0, & \text{if } \xi < 0; \\ 1, & \text{if } \xi \geq 0. \end{cases}$$

Let

$$\mathbb{H}^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}),\ \hat{f}(\xi) = 0, \forall \xi < 0\}.$$

Let I be the unit operator. Let H be the Hilbert transformation. It can be written as:

$$\widehat{Hf}(\xi) = -i\,\operatorname{sgn}(\xi)\hat{f}(\xi).$$

Then $P = \frac{1}{2}(I + iH)$ is the orthogonal projection operator from $L^2(\mathbb{R})$ to $H^2(\mathbb{R})$. Let

$$\hat{\tau}(\xi) = e^{-\frac{i}{2}\xi}\omega(\xi)\chi_{+}(\xi),\ \tau(x) = (2\pi)^{-1}\int_0^{\infty} e^{i(x-\frac{1}{2})\xi}\omega(\xi)d\xi.$$

For $j, k \in \mathbb{Z}$, denote $\tau_{j,k}(x) = 2^{\frac{j}{2}}\tau(2^j x - k)$. For $j \in \mathbb{Z}, k \in \mathbb{N}$, it is easy to see that $S_{j,k}(x) = \tau_{j,k}(x) + \tau_{j,-k-1}(x)$ are boundary limits of holomorphic functions. Recalling Paley-Wiener Theorem for bandlimited functions we know that $S_{j,k}$ are restrictions of entire functions of the Paley-Wiener type. We will call $S_{j,k}$ as holomorphic functions without ambiguity.

Lemma 3.3 *The set of holomorphic functions $\{S_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ forms an orthonormal basis of $\mathbb{H}^2(\mathbb{R})$.*

3.2.2 Holomorphic Wavelets on Interval

We will study $h_j(x)$ and their integrals, and then present a Lemma. $\omega(0) = 0$. For $j \geq 1$, denote

$$\begin{aligned} h_j(x) &= 2^{\frac{j}{2}}\sum_{l=-\infty}^{+\infty}\{\tau(2^j x - 2^j l) + \tau(2^j x + 2^j l + 1)\} \\ &= (2\pi)^{-1}2^{-\frac{j}{2}}\sum_{l=1}^{\infty}\omega(2l\pi 2^{-j})e^{2l\pi i(x-2^{-j-1})}, \\ \tilde{h}_j(x) &= \frac{1}{i\pi}2^{-\frac{3j}{2}}\sum_{l=1}^{\infty}\frac{w(2l\pi 2^{-j})}{2l\pi 2^{-j}}e^{2l\pi i(x-2^{-j-1})}. \end{aligned}$$

By the construction of $h_j(x)$ and $\tilde{h}_j(x)$, we have

$$(\tilde{h}_j(x))' = h_j(x). \tag{3.1}$$

From Meyer [9] and Qian-Yang [16], by wavelet property, we have

Lemma 3.4 $\forall N > 4$, $|x| \leq \frac{1}{2}$ *and* $j \geq 1$, *we have*

$$\begin{aligned} |h_j(x)| &\leq C_N 2^{\frac{j}{2}}(1 + |2^j x|)^{-N}, \\ |\tilde{h}_j(x)| &\leq C_N 2^{-\frac{j}{2}}(1 + |2^j x|)^{-N}. \end{aligned} \tag{3.2}$$

Further, we note that in the case

$$F(z) = \sum_{k \in \mathbb{N}} C_k z^k,$$

$F(z)$ is an analytic function in the unit disc. For $z = re^{2\pi ix}$, we may write $f(x) = F(e^{2\pi ix})$. For $j \geq 1$, let

$$h_j(x) = (2\pi)^{-1} 2^{-\frac{j}{2}} \sum_{l=1}^{\infty} \omega(2l\pi 2^{-j}) e^{2l\pi i(x - 2^{-j-1})}.$$

Let $G_0(x) = 1, G_1(x) = G_{0,0}(x) = e^{2\pi ix}$. $\forall j \geq 1, 0 \leq k < 2^{j-1}$, and $m = 2^{j-1} + k + 1, k^* = -k - 1$, define

$$\begin{aligned} G_m(x) = G_{j,k}(x) &= P(g_{j,k} + g_{j,k^*})(x) = h_j(x - k2^{-j}) \\ &= \frac{1}{\pi} 2^{-\frac{j}{2}} \sum_{1}^{\infty} w(2l\pi 2^{-j}) \cos(2l\pi(k + \tfrac{1}{2})2^{-j}) e^{2l\pi ix}. \end{aligned}$$

Let Λ_a denote the set $\{0\} \bigcup \{(j, k), j \in \mathbb{N}, 0 \leq k < 2^{j-1}\}$. We note that Λ_a is different from the index set Λ in Sect. 3.1.2 and $\Lambda_a \subsetneq \Lambda$. The holomorphic wavelets $\{G_m(x)\}_{m \geq 0}$ on the interval is different from the traditional wavelets on the interval in the real analysis. The following result is well-known. See [9].

Lemma 3.5 $\{G_m(x)\}_{m \in \mathbb{N}}$ *is an orthonormal basis of* $\mathbb{H}^2([0, 1])$.

3.3 Bimodal Wavelets for $\mathbb{H}^p(D)$

For $\mathbb{H}^p(D)(p \neq 2)$, [14] shows that TM systems are Schauder systems. To better understand the structure of the functions, we are to get some unconditional bases. Botchkariev [2] and Wojtaszczyk [22] applied Franklin system to analyze $\mathbb{H}^p(D)$. Afterwards, Meyer [9] used bimodal wavelets to restudy $\mathbb{H}^p(D)$. Bimodal wavelets are constructed using complex wavelet techniques, periodization of wavelets on lines and orthogonal projection. See Meyer [9] and Qian-Yang [16]. Let $G_0(z) = 1$ and

$G_1(z) = z$. By the Cauchy formula, for $m \geq 2, m \sim (j, k)$,

$$G_m(z) = \frac{1}{\pi} 2^{-\frac{j}{2}} \sum_{l=0}^{\infty} w(2l\pi 2^{-j}) \cos(2l\pi(k + \tfrac{1}{2})2^{-j}) z^l.$$

$\{G_m(z)\}_{m\geq 0}$ is an orthonormal basis of $\mathbb{H}^2(D)$. Let $f_0 = \langle F, G_0 \rangle$ and $f_1 = \langle F, G_1 \rangle$. For $m \geq 2$, let $f_m = f_{j,k} = \langle F, G_{j,k} \rangle = \langle F, G_m \rangle$. In combination with [2, 9, 22], the following result is known in [16]:

Lemma 3.6 Botchkariev-Meyer-Wojtaszcyk Theorem. *For* $1 < p < \infty$,

$$F = \sum_{m\geq 0} f_m G_m(z) \in \mathbb{H}^p(D)$$
$$\Leftrightarrow |f_0| + |f_1| + \int_0^{\frac{1}{2}} [\sum_{j\geq 1, 0\leq k < 2^{j-1}} 2^j |f_{j,k}|^2 \chi(2^j x - k)]^{\frac{p}{2}} dx < \infty.$$

Our idea is to generalize this result to the case of rational approximation in the disc. Later we will analyze the effect of our algorithm by comparison between the rational system and the Meyer wavelets.

4 Rational System

Liu-Yang-Yang and Yang-Chen have considered the boundary values of harmonic functions in [7, 25] which have some differences from analytic functions. Botchkariev [2] and Wojtaszczyk [22] applied Franklin system to analyze $\mathbb{H}^p(D)$. Meyer [9] used bimodal wavelets to restudy $\mathbb{H}^p(D)$. But rational functions are simple and intuitive, we extend their results to rational system.

4.1 Rational Functions $\{H_m\}_{m\geq 0}$ and Pseudo-orthogonality

4.1.1 Rational Functions

Let us first study the properties of rational functions. For $a = re^{2\pi i h}$, denote the a-parameterized Szegö kernel by $e_a(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$. The boundary value of $e_a(z)$ is

$$\frac{\sqrt{1-r^2}}{1-re^{2\pi i(x-h)}} = \frac{(1 - r\cos 2\pi(x-h))\sqrt{1-r^2}}{1 - 2r\cos 2\pi(x-h) + r^2} + i\frac{r \sin 2\pi(x-h)\sqrt{1-r^2}}{1 - 2r\cos 2\pi(x-h) + r^2}$$

The rational function is of unit L^2-norm on the circle which has no vanishing moment:

Lemma 4.1 *(i)* $\int_0^1 e_a dx = \sqrt{1-r^2}$.
(ii) $\langle e_a, e_a \rangle = 1$.

The conclusion (i) is essentially by the Cauchy formula; and (ii) by reducing to the integral of the Poisson kernel. We provide some useful computation.

Proof We prove (i) firstly.

$$\begin{aligned}\int_0^1 e_a dx &= \sqrt{1-r^2}\int_0^1\{\tfrac{(1-r\cos 2\pi(x-h))}{1-2r\cos 2\pi(x-h)+r^2} + i\tfrac{r\sin 2\pi(x-h))}{1-2r\cos 2\pi(x-h)+r^2}\}dx\\ &= \sqrt{1-r^2}\int_0^1 \tfrac{(1-r\cos 2\pi(x-h))}{1-2r\cos 2\pi(x-h)+r^2}dx.\end{aligned}$$

By substitution $y = 2\pi(x-h)$,

$$\begin{aligned}\int_0^1 e_a dx &= \tfrac{\sqrt{1-r^2}}{2\pi}\int_h^{2\pi-h}\tfrac{(1-r\cos y)}{1-2r\cos y+r^2}dy\\ &= \tfrac{\sqrt{1-r^2}}{2\pi}\int_0^{2\pi}\tfrac{(1-r\cos y)}{1-2r\cos y+r^2}dy\\ &= \tfrac{\sqrt{1-r^2}}{4\pi}\int_0^{2\pi}[1+\tfrac{(1-r^2)}{1-2r\cos y+r^2}]dy\\ &= \tfrac{\sqrt{1-r^2}}{4\pi}[\int_0^{2\pi}dy+\int_0^{2\pi}\tfrac{1-r^2}{1-2r\cos y+r^2}dy]\\ &= \tfrac{\sqrt{1-r^2}}{4\pi}[2\pi+2\pi] = \sqrt{1-r^2}.\end{aligned}$$

As for (ii), we have

$$\begin{aligned}\langle e_a, e_a\rangle &= \int_0^1 \tfrac{1}{1-\bar{a}z}\tfrac{1}{1-a\bar{z}}dx\\ &= (1-r^2)\int_0^1 \tfrac{1}{1-re^{2\pi i(x-h)}}\tfrac{1}{1-re^{-2\pi i(x-h)}}dx\\ &= \int_0^1 \tfrac{1-r^2}{1-2r\cos 2\pi(x-h)+r^2}dx.\end{aligned}$$

By substitution $y = 2\pi(x-h)$,

$$\begin{aligned}\langle e_a, e_a\rangle &= \int_0^1 \tfrac{1-r^2}{1-2r\cos 2\pi(x-h)+r^2}dx\\ &= \tfrac{1}{2\pi}\int_0^{2\pi}\tfrac{1-r^2}{1-2r\cos y+r^2}dy = 1.\end{aligned}$$

□

Denote $z = e^{2\pi i x}$, then $dz = 2\pi i z dx$. For the analytic Hardy space functions, by the Cauchy formula (or see page 18 of Qian's book [11]),

Lemma 4.2

$$\begin{aligned}\langle f, e_a\rangle &= \frac{\sqrt{1-|a|^2}}{2\pi}\int_0^{2\pi}\frac{f(e^{ix})}{1-ae^{-ix}}dx = \sqrt{1-|a|^2}\int_0^1\frac{f(e^{2\pi i x})}{1-ae^{-2\pi i x}}dx\\ &= \frac{\sqrt{1-|a|^2}}{2\pi i}\int_{\partial D}\frac{f(z)}{z-a}dz = \sqrt{1-|a|^2}f(a).\end{aligned}\tag{4.1}$$

Through taking differences between elements of the Szegö dictionary we create a system with zero moment. For $x \in \mathbb{R}$, let $[x]$ denote the maximum integer part. For $j \geq 1, 0 \leq k < 2^j$, denote

$$r_j = \sqrt{1-2^{-j}}, h_{j,k} = 2^{-j}k \quad \text{and} \quad a_{j,k} = r_j e^{2\pi i h_{j,k}}.$$

Hence $1 - r_j^2 = 2^{-j}$. We introduce to a minimum dictionary. The integral of the rational function is not zero, so we use the difference of the rational functions. Denote $H_0(z) = 1$, $H_1(z) = H_{0,0}(z) = z$. For $j \geq 1, 0 \leq k < 2^{j-1}$ and $m = 2^{j-1} + 1 + k$, denote $H_2(z) = H_{1,0}(z) = 2^{-\frac{1}{2}} - e_{a_{1,0}}$, $H_3(z) = H_{2,0}(z) = 2^{-\frac{1}{2}} e_{a_{1,0}} - e_{a_{2,0}}$, $H_4(z) = H_{2,1}(z) = 2^{-\frac{1}{2}} e_{a_{1,0}} - e_{a_{2,1}}, \cdots$. Generally, for $j \geq 2, 0 \leq k < 2^{j-1}$, denote $H_{2^{j-1}+1+k}(z) = H_{j,k}(z) = 2^{-\frac{1}{2}} e_{a_{j-1,[\frac{k}{2}]}} - e_{a_{j,k}}$. That is

$$H_m(z) = H_{j,k}(z) = 2^{-\frac{1}{2}} e_{a_{j-1,[\frac{k}{2}]}} - e_{a_{j,k}} = \sqrt{1 - r_j^2}\{\frac{1}{1 - \overline{a_{j-1,[\frac{k}{2}]}}z} - \frac{1}{1 - \overline{a_{j,k}}z}\}$$
$$= \sqrt{1 - r_j^2}\{\frac{(a_{j-1,[\frac{k}{2}]} - \overline{a_{j,k}})z}{(1 - \overline{a_{j-1,[\frac{k}{2}]}}z)(1 - \overline{a_{j,k}}z)}\}. \quad (4.2)$$

It is easy to see

Proposition 4.3 $\forall 1 < p < \infty, m \geq 0$, $H_m(z) \in \mathbb{H}^p(D)$.

4.1.2 Pseudo-orthogonality of Rational System

The above set of rational functions $\{H_m\}_{m \geq 0}$ have quasi-orthogonality. We know $\{H_m\}_{m \geq 1}$ all have zero vanishing moment. So we need only to consider $\{H_m\}_{m \geq 1} = \{H_{j,k}\}_{j \geq 0, 0 \leq k < 2^{j-1}}$. Let $\tilde{\tau}_{j,k,j',k'} = \langle H_{j,k}, H_{j',k'} \rangle$. Similar to the Eq. (3.1) of Proposition 1 in Chapter 8, Section 3, Volume 2 of Meyer's book [9], we have the following estimation:

Proposition 4.4 *For $j, j' \geq 0, 0 \leq k < 2^{j-1}, 0 \leq k' < 2^{j'-1}$, we have*

$$|\tilde{\tau}_{j,k,j',k'}| \leq C 2^{-\frac{3}{2}|j-j'|} \left(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |k 2^{-j} - k' 2^{-j'}|} \right)^2. \quad (4.3)$$

Proof We consider only $j, j' \geq 2, 0 \leq k < 2^{j-1}, 0 \leq k' < 2^{j'-1}$. By the inner product relation in equation (4.1), we have

$$\tilde{\tau}_{j,k,j',k'} = 2^{-\frac{1}{2}} \sqrt{1 - |a_{j'-1,[\frac{k'}{2}]}|^2} H_{j,k}(a_{j'-1,[\frac{k'}{2}]}) - \sqrt{1 - |a_{j',k'}|^2} H_{j,k}(a_{j',k'})$$
$$= \sqrt{1 - r_{j'}^2}\{H_{j,k}(a_{j'-1,[\frac{k'}{2}]}) - H_{j,k}(a_{j',k'})\}.$$

If $j = j'$, then, by similarity, we may restrict to the case $j \geq 2$ and $k = 0$ and $k' = 2l$. We have

$$|\tilde{\tau}_{j,0,j,2l}| = \sqrt{1 - r_{j'}^2} |H_{j,0}(a_{j'-1,l}) - H_{j,0}(a_{j',2l})|.$$

We estimate then $|H_{j,0}(a_{j-1,l})|$ and $|H_{j,0}(a_{j,2l})|$. The absolute value of $e^{2\pi i 2^{-j}2l}$ is 1, $r_{j-1} \sim 1$ and $r_j \sim 1$, we have

$$
\begin{aligned}
|H_{j,0}(a_{j-1,l})| &= \sqrt{1-r_j^2}\left|\frac{(r_{j-1}-r_j)r_{j-1}e^{2\pi i 2^{-j}2l}}{(1-r_{j-1}^2e^{2\pi i 2^{-j}2l})(1-r_{j-1}r_je^{2\pi i 2^{-j}2l})}\right| \\
&\leq \sqrt{1-r_j^2}(r_{j-1}-r_j)\frac{1}{|1-r_{j-1}^2e^{2\pi i 2^{-j}2l}|\cdot|1-r_{j-1}r_je^{2\pi i 2^{-j}2l}|}, \\
|H_{j,0}(a_{j,2l})| &\leq \sqrt{1-r_j^2}(r_{j-1}-r_j)\frac{1}{|1-r_{j-1}r_je^{2\pi i 2^{-j}2l}|\cdot|1-r_j^2e^{2\pi i 2^{-j}2l}|}.
\end{aligned}
$$

We first estimate the term $r_{j-1}-r_j$:

$$
r_{j-1}-r_j = \frac{r_{j-1}^2-r_j^2}{r_{j-1}+r_j} \sim \frac{1}{2}2^{-j}.
$$

We now compute the fractional terms in $|H_{j,0}(a_{j-1,l})|$ and $|H_{j,0}(a_{j,2l})|$ that have absolute values. Bcause $1-\cos(2\pi 2^{-j}2l) \sim 8\pi^2 2^{-2j}l^2$, we have

$$
\begin{aligned}
\frac{1}{|1-r_{j-1}^2e^{2\pi i 2^{-j}2l}|} &\leq \frac{1}{\sqrt{(1-r_{j-1}^2\cos(2\pi 2^{-j}2l))^2+(r_{j-1}^2\sin(2\pi 2^{-j}2l))^2}} \\
&= \frac{1}{\sqrt{(1-r_{j-1}^2)^2+2r_{j-1}^2(1-\cos(2\pi 2^{-j}2l))}} \\
&\leq \frac{1}{\sqrt{(2\cdot 2^{-j})^2+16\pi^2 2^{-2j}l^2}} \\
&= \frac{1}{2\cdot 2^{-j}\sqrt{1+4\pi^2 l^2}}.
\end{aligned}
$$

Similarly,

$$
\frac{1}{|1-r_j^2e^{2\pi i 2^{-j}2l}|} \leq \frac{1}{\sqrt{2^{-2j}+16\pi^2 2^{-2j}l^2}} = \frac{1}{2\cdot 2^{-j}\sqrt{\frac{1}{4}+4\pi^2 l^2}}.
$$

Since $(1-r_{j-1}r_j)^2 = (\frac{1-r_{j-1}^2r_j^2}{1+r_{j-1}r_j})^2 \sim (\frac{3}{2}2^{-j}-2^{-2j})^2 \sim \frac{9}{4}2^{-2j}$, we have

$$
\frac{1}{|1-r_{j-1}r_je^{2\pi i 2^{-j}2l}|} \leq \frac{1}{\sqrt{\frac{9}{4}2^{-2j}+16\pi^2 2^{-2j}l^2}} = \frac{1}{2\cdot 2^{-j}\sqrt{\frac{9}{16}+4\pi^2 l^2}}.
$$

Applying the above estimations to $|\tilde{\tau}_{j,0,j,2l}|$, we get

$$
\begin{aligned}
|\tilde{\tau}_{j,0,j,2l}| &\leq \sqrt{1-r_j^2}(|H_{j,0}(a_{j-1,l})|+|H_{j,0}(a_{j,2l})|) \\
&\leq \frac{(1-r_j^2)(r_{j-1}-r_j)}{4\cdot 2^{-2j}}\frac{1}{\sqrt{\frac{9}{16}+4\pi^2 l^2}}\left(\frac{1}{\sqrt{1+4\pi^2 l^2}}+\frac{1}{\sqrt{\frac{1}{4}+4\pi^2 l^2}}\right) \\
&\leq \frac{C}{1+16\pi^2 l^2}.
\end{aligned}
$$

If $j > j'$, by similarity, we may restrict to the case $k = 0$, $k' = 2l$. In this case,

$$|\tilde{\tau}_{j,0,j',2l}| = \sqrt{1 - r_{j'}^2}|H_{j,0}(a_{j'-1,l}) - H_{j,0}(a_{j',2l})|.$$

Then we consider the estimations of $|H_{j,0}(a_{j'-1,l})|$ and $|H_{j,0}(a_{j',2l})|$. We note that the absolute value of $e^{2\pi i 2^{-j'}2l}$ is 1, $r_{j-1} \sim 1$ and $r_{j'} \sim 1$. Hence,

$$\begin{aligned}
|H_{j,0}(a_{j'-1,l})| &= \sqrt{1 - r_j^2}\left|\frac{(r_{j-1}-r_j)r_{j-1}e^{2\pi i 2^{-j'}2l}}{(1-r_{j-1}^2 e^{2\pi i 2^{-j'}2l})(1-r_{j-1}r_j e^{2\pi i 2^{-j'}2l})}\right| \\
&\leq \sqrt{1 - r_j^2}(r_{j-1} - r_j)\frac{1}{|1-r_{j-1}r_{j'-1}e^{2\pi i 2^{-j'}2l}|\cdot|1-r_{j'-1}r_j e^{2\pi i 2^{-j'}2l}|}, \\
|H_{j,0}(a_{j',2l})| &\leq \sqrt{1 - r_j^2}(r_{j-1} - r_j)\frac{1}{|1-r_{j-1}r_{j'}e^{2\pi i 2^{-j'}2l}|\cdot|1-r_j r_{j'} e^{2\pi i 2^{-j'}2l}|}.
\end{aligned}$$

We estimate first the fractional parts in the above inequalities.

$$\begin{aligned}
\frac{1}{|1-r_{j-1}r_{j'-1}e^{2\pi i 2^{-j'}2l}|} &\leq \frac{1}{\sqrt{(1-r_{j-1}r_{j'-1})^2+2r_{j-1}r_{j'-1}(1-\cos(2\pi 2^{-j'}2l))}} \\
&\leq 2^{j'}\frac{1}{\sqrt{16\pi^2 l^2+(1+2^{j'-j})^2}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{|1 - r_j r_{j'-1}e^{2\pi i 2^{-j'}2l}|} &\leq 2^{j'}\frac{1}{\sqrt{16\pi^2 l^2 + (1 + 2^{j'-j-1})^2}}, \\
\frac{1}{|1 - r_{j-1}r_{j'}e^{2\pi i 2^{-j'}2l}|} &\leq 2^{j'}\frac{1}{\sqrt{16\pi^2 l^2 + (\frac{1}{2} + 2^{j'-j})^2}},
\end{aligned}$$

and

$$\frac{1}{|1 - r_j r_{j'}e^{2\pi i 2^{-j'}2l}|} \leq 2^{j'}\frac{1}{\sqrt{16\pi^2 l^2 + (\frac{1}{2} + 2^{j'-j-1})^2}}.$$

Applying then the above estimations to $|\tilde{\tau}_{j,0,j',2l}|$, we have

$$\begin{aligned}
|\tilde{\tau}_{j,0,j',2l}| &\leq \sqrt{1 - r_{j'}^2}(|H_{j,0}(a_{j'-1,l})| + |H_{j,0}(a_{j',2l})|) \\
&\leq \frac{1}{2}2^{-\frac{3}{2}(j-j')}\frac{1}{\sqrt{16\pi^2 l^2+(1+2^{j'-j-1})^2}}\left\{\frac{1}{\sqrt{16\pi^2 l^2+(1+2^{j'-j})^2}} + \frac{1}{\sqrt{16\pi^2 l^2+(\frac{1}{2}+2^{j'-j})^2}}\right\} \\
&\leq 2^{-\frac{3}{2}(j-j')}\frac{C}{1+16\pi^2 l^2}.
\end{aligned}$$

□

4.1.3 Pseudo-orthogonality Between $\{H_m\}$ and Meyer Wavelets $\{G_m\}$

Let $H_{j,k}$ be the functions defined in Sect. 4.1 which is based on TM system. Let $G_{j,k}$ be the orthogonal bimodal wavelets induced by Meyer's algorithm. Functions $\{H_m\}_{m\geq 0}$ are comparable to bimodal wavelets $\{G_m\}_{m\geq 0}$. Let $\tau_{j,k,j',k'} = \langle G_{j,k}, H_{j',k'}\rangle$. Similar to the Eq. (3.1) of Proposition 1 in Section 3, Chapter 8, Volume 2 of Meyer's book [9], we have the following estimation:

Proposition 4.5 *For $j, j' \geq 0$, $0 \leq k < 2^{j-1}$, $0 \leq k' < 2^{j'-1}$, we have*

$$|\tau_{j,k,j',k'}| \leq C2^{-\frac{3}{2}|j-j'|}\left(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|}\right)^2. \tag{4.4}$$

Proof Let's first rewrite the expression for $\tau_{j,k,j',k'}$. We use substitution to turn $x - 2^{-j}k$ into x and get

$$\tau_{j,k,j',k'} = \langle G_{j,k}, H_{j',k'}\rangle = \langle h_j(x), H_{j',k'}(x+2^{-j}k)\rangle. \tag{4.5}$$

Denote $D_{j',k'}(x) = (H_{j',k'}(x))'$ and by using integration by parts, we have

$$|\tau_{j,k,j',k'}| = |\langle \tilde{h}_j(x), D_{j',k'}(x+2^{-j}k)\rangle|. \tag{4.6}$$

Next, we estimate $H_{j',k'}(x)$ and $D_{j',k'}(x)$. By similarity, we consider only $k' = 2\alpha$, $\alpha \in \mathbb{N}$. We have

$$H_{j',2\alpha}(x) = 2^{-\frac{1}{2}}e_{a_{j'-1,\alpha}} - e_{a_{j',2\alpha}} = 2^{-\frac{j'}{2}}\left\{\frac{1}{1-r_{j'-1}e^{2\pi i(x-h_{j',2\alpha})}} - \frac{1}{1-r_{j'}e^{2\pi i(x-h_{j',2\alpha})}}\right\}.$$

Hence

$$\begin{aligned}
|H_{j',2\alpha}(x)| &\leq 2^{-\frac{j'}{2}}\left|\frac{1}{1-r_{j'-1}e^{2\pi i(x-h_{j',2\alpha})}} - \frac{1}{1-r_{j'}e^{2\pi i(x-h_{j',2\alpha})}}\right| \\
&\leq C2^{-\frac{3j'}{2}}(2^{-j'}+|x-2^{1-j'}\alpha|)^{-2} \leq C2^{\frac{j'}{2}}(1+|2^{j'}x-2\alpha|)^{-2}, \\
|D_{j',2\alpha}(x)| &\leq C2^{\frac{3j'}{2}}(1+|2^{j'}x-2\alpha|)^{-2}.
\end{aligned} \tag{4.7}$$

Below, we estimate $|\tau_{j,k,j',k'}|$ in two cases. (I) For $j \leq j'$, by similarity, we may restrict to the case $j, j' \geq 2$ and $k' = 2\alpha$. By Lemma 3.4, equations (4.5) and (4.7), we have

$$\begin{aligned}
|\tau_{j,k,j',2\alpha}| &\leq \int_0^1 |h_j(x)||H_{j',2\alpha}(x+2^{-j}k)|dx \\
&\leq C2^{\frac{j+j'}{2}}\int_0^1 (1+|2^jx|)^{-N}(1+|2^{j'}x+2^{j'-j}k-2\alpha|)^{-2}dx.
\end{aligned}$$

We distinguish two cases: (i) $|2^{j'}x| < \frac{1}{2}|2^{j'-j}k - 2\alpha|$ and (ii) $|2^{j'}x| \geq \frac{1}{2}|2^{j'-j}k - 2\alpha|$. We get

$$|\tau_{j,k,j',2\alpha}| \leq C2^{\frac{3(j-j')}{2}}(1 + |k - 2\alpha 2^{j-j'}|)^{-2}.$$

(II) For $j > j'$, by similarity, we may restrict to the case $j, j' \geq 2$ and $k' = 2\alpha$. Similarly, by Lemma 3.4, equations (4.6) and (4.7), we have

$$\begin{aligned}|\tau_{j,k,j',2\alpha}| &\leq \int_0^1 |\tilde{h}_j(x)||D_{j',k'}(x + 2^{-j}k)|dx \\ &\leq C2^{\frac{-j+3j'}{2}} \int_0^1 (1 + |2^j x|)^{-N}(1 + |2^{j'}x + 2^{j'-j}k - 2\alpha|)^{-2}dx.\end{aligned}$$

We distinguish two cases: (i) $|2^{j'}x| < \frac{1}{2}|2^{j'-j}k - 2\alpha|$ and (ii) $|2^{j'}x| \geq \frac{1}{2}|2^{j'-j}k - 2\alpha|$. We get

$$|\tau_{j,k,j',2\alpha}| \leq 2^{-\frac{3}{2}|j-j'|} \frac{C}{(1 + |k2^{j'-j} - 2\alpha|)^2}.$$

□

4.2 Maximum Operator and Hardy Spaces

By the above rational system, we can define $\tilde{\mathbb{H}}^p(D)$ by the following Lusin area integral. In this subsection, we prove that $\tilde{\mathbb{H}}^p(D) \subset \mathbb{H}^p(D)$. In the next section, we prove the completeness of our rational system $\{H_m\}_{m\geq 0}$ in Hardy space $\mathbb{H}^p(D)$. Hence $\tilde{\mathbb{H}}^p(D)$ is just Hardy space $\mathbb{H}^p(D)$.

Definition 4.6 Denote formally by $F \sim f_0 + f_1 + \sum_{j\geq 1, 0\leq k<2^{j-1}} f_{j,k} H_{j,k}$. We say that the form $F \in \tilde{\mathbb{H}}^p(D)$, if

$$|f_0| + |f_1| + \int_0^{\frac{1}{2}} \left[\sum_{j\geq 1, 0\leq k<2^{j-1}} 2^j |f_{j,k}|^2 \chi(2^j x - k)\right]^{\frac{p}{2}} dx < \infty.$$

For $j \geq 1$, denote

$$f_j(x) = \sum_{0\leq k<2^{j-1}} 2^{\frac{j}{2}} |f_{j,k}| \chi(2^j x - k). \tag{4.8}$$

We use Hardy-Littlewood maximum operator to consider Hardy spaces $\mathbb{H}^p(D)$. See [3, 20, 26]. Hardy-Littlewood maximum operator in $\mathbb{R}$ is defined as follows:

$$M_{\mathbb{R}} f(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(x)|dx.$$

Hardy Littlewood maximum operator on the interval is denoted

$$Mf(x) = \sup_{x \in Q, |Q| \leq 1} \frac{1}{|Q|} \int_Q |f(x)| dx.$$

The $\mathbb{H}^p$ type space is the space of type $L^p(l^2)$, see [3, 20]. Further, the norm comparison between them is realized by the following Fefferman-Stein vector value maximum operator theorem. See [5].

Lemma 4.7 *For* $1 < p, q < \infty$, *then*

$$\|\{Mf_k\}\|_{L^p(l^q)} \leq C\|\{f_k\}\|_{L^p(l^q)}, \forall f = \{f_k\}.$$

Now we conclude that the relative $\tilde{\mathbb{H}}^p$ induced in [21] in is a subspace of the $\mathbb{H}^p$ induced by Meyer's algorithm. According to Fefferman-Stein vector value maximum operator theorem and the results in section 2 and section 11 in chapetr 6 of Meyer [9], we have the following Botchkariev-Meyer-Wojtaszczyk Theorem for rational system:

Proposition 4.8 *If* $F \sim \sum_{m \geq 0} f_m H_m \in \tilde{\mathbb{H}}^p(D)$, *then* $F \in \mathbb{H}^p(D)$.

Proof Since $\{G_m\}_{m \geq 0}$ is an unconditional basis in $\mathbb{H}^p(D)$ (see [9, 16]), hence

$$F = \sum_{m'} \sum_{m \geq 0} f_m \langle H_m, G_{m'} \rangle G_{m'} = \sum_{m'} \sum_{m \geq 0} f_m \tau_{m,m'} G_{m'}$$

is true in the sense of distribution. We write them with double indices,

$$F = f_0 G_0 + \sum_{j' \geq 0, 0 \leq k' < 2^{j'-1}} \sum_{j \geq 0, 0 \leq k < 2^{j-1}} f_{j,k} \tau_{j,k,j',k'} G_{j',k'}.$$

By Lemma 3.6, we get

$$\|F\|_{\mathbb{H}^p} \leq |f_0| + C\|[\sum_{j' \geq 0, 0 \leq k' < 2^{j'-1}} 2^{j'}(\sum_{j \geq 0, 0 \leq k < 2^{j-1}} f_{j,k} \tau_{j,k,j',k'})^2 \chi(2^{j'}x - k')]^{\frac{1}{2}}\|_{L^p}.$$

For f_j defined in the equation (4.8), by the definition of Hardy Littlewood maximum operator on the interval, we have

$$\begin{aligned} Mf_j(x) &= \sup_{|Q| \leq 1} \tfrac{1}{|Q|} \textstyle\int_Q |f_j(x)| dx \\ &= \sup_{x, |r| \leq 1} \tfrac{1}{|r|} \sum_{-r - 2^{-j} \leq 2^{-j}k - x \leq r} 2^{-\frac{j}{2}} |f_{j,k}|. \end{aligned}$$

For $j \geq j' \geq 0$ and $\tau = 1$, denote $g^{1,j}_{j',k'} = 2^{\frac{j'}{2}} \sum_{|k2^{j'-j}-k'|<2} f_{j,k}\tau_{j,k,j',k'}$. For $\tau \geq 2$, $g^{\tau,j}_{j',k'} = 2^{\frac{j'}{2}} \sum_{2^{\tau-1}\leq|k2^{j'-j}-k'|<2^{\tau}} f_{j,k}\tau_{j,k,j',k'}$. Hence we can write $\sum_{0\leq k<2^{j-1}} f_{j,k}\tau_{j,k,j',k'} = \sum_{\tau\geq1} g^{\tau,j}_{j',k'}$. Denote $g_{j,j'} = \sum_{\tau\geq1} g^{\tau,j}_{j',k'}$ and

$$T = [\sum_{j'\geq0,0\leq k'<2^{j'-1}} 2^{j'}(\sum_{j\geq0,0\leq k<2^{j-1}} f_{j,k}\tau_{j,k,j',k'})^2\chi(2^{j'}x-k')]^{\frac{1}{2}} = \{\sum_{j'\geq0}|\sum_{j\geq0} g_{j,j'}|^2\}^{\frac{1}{2}}.$$

We first consider the estimate of $|g^{\tau,j}_{j',k'}|$.

$$\begin{aligned}|g^{\tau,j}_{j',k'}| &\leq C2^{\frac{j'}{2}}2^{-\frac{3}{2}|j-j'|}2^{-2\tau-2}\sum_{|2^{-j}k-2^{-j'}k'|\leq2^{\tau-j}}|f_{j,k}| \\ &\leq C2^{-|j-j'|}2^{-\tau-2}|Mf_j(x)|.\end{aligned}$$

Then, we have

$$\sum_{j\geq0}|g_{j,j'}| = \sum_{j\geq0}\sum_{\tau\geq1}|g^{\tau,j}_{j',k'}| \leq \sum_{j\geq0}\frac{C}{4}2^{|j-j'|}|Mf_j(x)|.$$

Thus,

$$\begin{aligned}T &= \left\{\sum_{j'\geq0}|\sum_{j\geq0}\frac{C}{4}2^{|j-j'|}|Mf_j(x)||^2\right\}^{\frac{1}{2}} \\ &\leq \left\{\sum_{j'\geq0}|(\sum_{j\geq0}\frac{C^2}{16}2^{|j-j'|})(\sum_{j\geq0}2^{|j-j'|}|Mf_j(x)|^2)|\right\}^{\frac{1}{2}} \\ &= \frac{C}{4}\left\{\sum_{j'\geq0}\sum_{j\geq0}2^{|j-j'|}|Mf_j(x)|^2\right\}^{\frac{1}{2}}.\end{aligned}$$

For $0 \leq j < j'$ and $\tau = 1$, denote $g^{1,j}_{j',k'} = 2^{\frac{j'}{2}} \sum_{|k-k'2^{j-j'}|<2} f_{j,k}\tau_{j,k,j',k'}$. For $\tau \geq 2$, $g^{\tau,j}_{j',k'} = 2^{\frac{j'}{2}} \sum_{2^{\tau-1}\leq|k-k'2^{j-j'}|<2^{\tau}} f_{j,k}\tau_{j,k,j',k'}$. Hence we can write $\sum_{0\leq k<2^{j-1}} f_{j,k}\tau_{j,k,j',k'} = \sum_{\tau\geq1} g^{\tau,j}_{j',k'}$. We first consider the estimate of $|g^{\tau,j}_{j',k'}|$.

$$\begin{aligned}|g^{\tau,j}_{j',k'}| &\leq C2^{\frac{j'}{2}}2^{-\frac{3}{2}|j-j'|}2^{-2\tau}\sum_{|2^{-j}k-2^{-j'}k'|\leq2^{\tau-j}}|f_{j,k}| \\ &\leq C2^{\frac{j'-j}{2}}2^{-\frac{3}{2}|j-j'|}2^{-\tau}\sum_{|2^{-j}k-2^{-j'}k'|\leq2^{\tau-j}}2^{-\frac{j}{2}}2^{j-\tau}|f_{j,k}| \\ &\leq C2^{-|j-j'|}2^{-\tau}|Mf_j(x)|.\end{aligned}$$

Then, we have

$$\sum_{j\geq0}|g_{j,j'}| = \sum_{j\geq0}\sum_{\tau\geq1}|g^{\tau,j}_{j',k'}| \leq \sum_{j\geq0}C2^{|j-j'|}|Mf_j(x)|.$$

Thus,

$$
\begin{aligned}
T &= \left\{ \sum_{j'\geq 0} | \sum_{j\geq 0} g_{j,j'} |^2 \right\}^{\frac{1}{2}} = \left\{ \sum_{j'\geq 0} | \sum_{j\geq 0} C 2^{|j-j'|} |Mf_j(x)||^2 \right\}^{\frac{1}{2}} \\
&\leq \left\{ \sum_{j'\geq 0} |(\sum_{j\geq 0} C^2 2^{|j-j'|})(\sum_{j\geq 0} 2^{|j-j'|} |Mf_j(x)|^2)| \right\}^{\frac{1}{2}} \\
&= C \left\{ \sum_{j'\geq 0} \sum_{j\geq 0} 2^{|j-j'|} |Mf_j(x)|^2 \right\}^{\frac{1}{2}}.
\end{aligned}
$$

By psedo-orthogonality or by the estimation of the decay of $\tau_{j,k,j',k'}$,

$$
\|F\|_{\mathbb{H}^p} \leq |f_0| + \| \left(\sum_{j'\geq 0} \sum_{j} 2^{-|j-j'|} |Mf_j|^2 \right)^{\frac{1}{2}} \|_{L^p} \leq |f_0| + \| \left(\sum_{j\geq 0} |Mf_j|^2 \right)^{\frac{1}{2}} \|_{L^p}.
$$

By applying Fefferman-Stein vector value maximum operator Lemma 4.7, we get the desired conclusion. □

The above Proposition implies the independence of linearity of $\{H_m\}_{m\geq 0}$:

Corollary 4.9

$$
\left\langle z^m, \sum_{m'\geq 0} h_{m'} H_{m'} \right\rangle = 0, \forall m \geq 0 \Rightarrow \forall m' \geq 0, h_{m'} = 0.
$$

5 Completeness of Basis

It is well known that a sufficient and necessary condition for TM system to become a basis is that the TM system satisfies the hyperbolic non-separability condition:

$$
\sum_{k=1}^{+\infty} (1 - |a_k|) = \infty.
$$

See [11–15, 17, 24]. In Proposition 5.2, we use properties of the matrices to prove that our rational system is a basis and get the completeness of our rational system. The completeness of the basis $\{H_m\}_{m\geq 0}$ has relation to the following matrix

$$
M = \begin{pmatrix} 1 & r_1 & r_1^2 & r_1^3 & \cdots \\ 1 & r_2 & r_2^2 & r_2^3 & \cdots \\ 1 & r_3 & r_3^2 & r_3^3 & \cdots \\ 1 & r_4 & r_4^2 & r_4^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{5.1}
$$

The above matrix is related to the following Vandermonde determinant D_n:

$$D_n = \begin{vmatrix} 1 & r_1 & r_1^2 & r_1^3 & \cdots & r_1^{n-1} \\ 1 & r_2 & r_2^2 & r_2^3 & \cdots & r_2^{n-1} \\ 1 & r_3 & r_3^2 & r_3^3 & \cdots & r_3^{n-1} \\ 1 & r_4 & r_4^2 & r_4^3 & \cdots & r_4^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1, & r_n & r_n^2 & r_n^3 & \cdots & r_n^{n-1} \end{vmatrix}. \tag{5.2}$$

The following Lemma is well-known:

Lemma 5.1 $D_n = \prod_{1 \le m < m' \le n} (r_m - r_{m'})$.

Now we consider the completeness of the basis $\{H_m\}_{m \ge 0}$.

Proposition 5.2

$$\left\langle \sum_{m' \ge 0} h_{m'} z^{m'}, H_m \right\rangle = 0, \forall m \ge 0 \Rightarrow \forall m' \ge 0, h_{m'} = 0.$$

Proof For $m = 2^{j-1} + 1 + k \ge 3$, by the definition $H_m = H_{j,k} = 2^{-\frac{1}{2}} e_{a_{j-1,[\frac{k}{2}]}} - e_{a_{j,k}}$ in equation (4.2), we have

$$\left\langle \sum_{m' \ge 2} h_{m'} z^{m'}, H_{j,k} \right\rangle = \sum_{m' \ge 2} h_{m'} \langle z^{m'}, H_{j,k} \rangle.$$

For $j \ge 1$, we first consider the expression for the inner product $\langle z^{m'}, H_{j,k} \rangle$ by the odd-even property of k.

If $k = 2l$, by definition, the above quantity is

$$\begin{aligned} \langle z^{m'}, H_m \rangle &= 2^{-\frac{1}{2}} \sqrt{1 - r_{j-1}^2}\, r_{j-1}^{m'} e^{m' 2\pi i h_{j-1,[\frac{k}{2}]}} - \sqrt{1 - r_j^2}\, r_j^{m'} e^{m' 2\pi i h_{j,k}} \\ &= \sqrt{1 - r_j^2} (r_{j-1}^{m'} e^{m' 2\pi i h_{j-1,[\frac{k}{2}]}} - r_j^{m'} e^{m' 2\pi i h_{j,k}}). \end{aligned}$$

If $k = 2l + 1$, by definition, the above quantity is

$$\begin{aligned} \langle z^{m'}, H_m \rangle &= 2^{-\frac{1}{2}} \sqrt{1 - r_{j-1}^2}\, r_{j-1}^{m'} e^{m' 2\pi i h_{j-1,[\frac{k}{2}]}} - \sqrt{1 - r_j^2}\, r_j^{m'} e^{m' 2\pi i h_{j,k}} \\ &= \sqrt{1 - r_j^2} (r_{j-1}^{m'} e^{m' 2\pi i h_{j-1,[\frac{k}{2}]}} - r_j^{m'} e^{m' 2\pi i h_{j,k}}). \end{aligned}$$

(i) For $H_m = H_{j,k} = 2^{-\frac{1}{2}} e_{a_{j-1,[\frac{k}{2}]}} - e_{a_{j,k}}$ in equation (4.2), we prove then

$$\left\langle \sum_{m' \ge 0} h_{m'} z^{m'}, H_{j,k} \right\rangle = 0, \forall j \ge 1, 0 \le k < 2^{j-1}$$

is equivalent to the following equation:

$$\left\langle \sum_{m'\geq 0} h_{m'} z^{m'}, a_{j,k} \right\rangle = 0, \forall j \geq 1, 0 \leq k < 2^{j-1}.$$

It is easy to see that $h_0 = 0$ and $h_1 = 0$. Hence, we consider the case of $m' \geq 2$ and we proved it by mathematical induction. For $j = 1$ and $k = 0$, we have

$$\begin{aligned} \langle \sum_{m'\geq 2} h_{m'} z^{m'}, H_m \rangle &= \sum_{m'\geq 2} h_{m'} \sqrt{1 - r_1^2}(r_0^{m'} e^{m' 2\pi i h_{0,0}} - r_1^{m'} e^{m' 2\pi i h_{1,0}}) \\ &= \sum_{m'\geq 2} h_{m'} \sqrt{1 - r_1^2}(r_0^{m'} - r_1^{m'}) \\ &= - \sum_{m'\geq 2} h_{m'} \sqrt{1 - r_1^2} r_1^{m'}. \end{aligned}$$

Because $\langle \sum_{m'\geq 2} h_{m'} z^{m'}, H_{1,0} \rangle = 0$, we get

$$\sum\nolimits_{m'\geq 2} h_{m'} \sqrt{1 - r_1^2}\, r_1^{m'} = 0.$$

For $j = 2$ and $k = 0$, we have

$$\begin{aligned} \left\langle \sum_{m'\geq 2} h_{m'} z^{m'}, H_{2,0} \right\rangle &= \sum_{m'\geq 2} h_{m'} \sqrt{1 - r_2^2}(r_1^{m'} e^{m' 2\pi i h_{1,0}} - r_2^{m'} e^{m' 2\pi i h_{2,0}}) \\ &= \sum_{m'\geq 2} h_{m'} \sqrt{1 - r_2^2}(r_1^{m'} - r_2^{m'}). \end{aligned}$$

Since $\langle \sum_{m'\geq 2} h_{m'} z^{m'}, H_{2,0} \rangle = 0$ and $\sum_{m'\geq 2} h_{m'} \sqrt{1 - r_1^2}\, r_1^{m'} = 0$, we get

$$\sum_{m'\geq 2} h_{m'} \sqrt{1 - r_2^2}\, r_2^{m'} = 0.$$

By induction and similarity, we get the following equation

$$\sum_{m'\geq 2} h_{m'} \sqrt{1 - r_j^2} r_j^{m'} e^{m' 2\pi i h_{j,k}} = 0, \forall j \geq 1, 0 \leq k < 2^{j-1}.$$

That is to say, $\forall j \geq 1, 0 \leq k < 2^{j-1}$, we have

$$\left\langle \sum_{m'\geq 2} h_{m'} z^{m'}, a_{j,k} \right\rangle = \sum_{m'\geq 2} h_{m'} \sqrt{1 - r_j^2} r_j^{m'} e^{m' 2\pi i h_{j,k}} = 0.$$

The above equality implies

$$\sum_{m'\geq 2} h_{m'} r_j^{m'} e^{m' 2\pi i h_{j,k}} = 0, \forall j \geq 1, 0 \leq k < 2^{j-1}. \tag{5.3}$$

(ii) Then we can write the above equation (5.3) as a system of equations:

$$\begin{aligned}
j = 1, \quad & \sum_{m'\geq 2} h_{m'} r_1^{m'} = 0; \\
j = 2, \quad & \sum_{m'\geq 2} h_{m'} r_2^{m'} = 0; \\
& \sum_{m'\geq 2} h_{m'} r_2^{m'} e^{\frac{\pi}{2} i m'} = 0; \\
j = 3, \quad & \sum_{m'\geq 2} h_{m'} r_3^{m'} = 0; \\
& \sum_{m'\geq 2} h_{m'} r_3^{m'} e^{\frac{\pi}{4} i m'} = 0; \\
& \sum_{m'\geq 2} h_{m'} r_3^{m'} e^{\frac{\pi}{2} i m'} = 0; \\
& \sum_{m'\geq 2} h_{m'} r_3^{m'} e^{\frac{3\pi}{4} i m'} = 0; \\
j = 4, \quad & \sum_{m'\geq 2} h_{m'} r_4^{m'} = 0; \\
& \cdots
\end{aligned} \tag{5.4}$$

In the above equations (5.4), we consider the first equation for each $j \geq 1$, we get

$$\begin{aligned}
& \sum_{m'\geq 2} h_{m'} r_1^{m'} = 0; \\
& \sum_{m'\geq 2} h_{m'} r_2^{m'} = 0; \\
& \sum_{m'\geq 2} h_{m'} r_3^{m'} = 0; \\
& \sum_{m'\geq 2} h_{m'} r_4^{m'} = 0; \\
& \cdots
\end{aligned} \tag{5.5}$$

Denote the transpose vector of $(h_2, h_3, h_4, \cdots)$ be $(h_2, h_3, h_4, \cdots)^T$. We divide by r_j^2 in the above (5.5), we get the matrix M in the equation (5.1) and

$$M \cdot (h_2, h_3, h_4, \cdots)^T = 0.$$

Combine Vandermonde determinant (5.2) with above equation, we get

$$\forall m' \geq 2, h_{m'} = 0.$$

□

6 Rearrangement of Partial Sum Order and Adaptive Expansion

There is an ample amount of literature studying matching pursuit algorithms including the type based on dictionary and basis in which the branch in relation to TM systems is called adaptive Fourier decomposition (AFD). See [11–15, 17, 24]. The present study is also based on TM system and can be viewed as a variation of the AFD methods. What we do is we replace the orthogonal basis with the bi-orthogonal basis, and we replace the optimal matching pursuit with the partial maximum choice principle. Finally, we prove our main Theorem in the second subsection.

6.1 Maximal Partial Sum Re-ordering

First, we construct a dual basis to replace the orthogonal basis in the traditional AFD algorithm. For $m, m' \geq 0$, let $\delta_{m,m'} = 1$, if $m = m'$ and $\delta_{m,m'} = 0$, if $m \neq m'$.

Definition 6.1 Given $1 < p < \infty$ and $p' = \frac{p}{p-1}$. For $m \geq 0$, we assume that $H_m \in H^p(D)$ and $\tilde{H}_m \in H^{p'}(D)$. Let $\{H_m\}_{m\geq 0}$ be a basis of $H^p(D)$. We say that $\{\tilde{H}_m\}_{m\geq 0}$ is a dual basis of $\{H_m\}_{m\geq 0}$, if

$$\langle H_m, \tilde{H}_{m'} \rangle = \delta_{m,m'}, \forall m, m' \geq 0.$$

For a given $\{H_m\}_{m\geq 0}$ in $\mathbb{H}^p(D)$, we can look for a dual basis $\{\tilde{H}_m\}_{m\geq 0}$ in the sense of functional way where $\tilde{H}_m \in \mathbb{H}^{p'}(D)$. That is,

(1) Take $\tilde{H}_0 = 1 = H_0$, $\tilde{H}_1 = \tilde{H}_{0,0} = e^{2\pi i x} = H_{0,0} = H_1$.
(2) Take $\tilde{H}_2 \in H^{p'}(D)$ such that

$$\langle \tilde{H}_2, H_m \rangle = \delta_{m,2}.$$

(3) Take $\tilde{H}_3 \in H^{p'}(D)$ such that

$$\langle \tilde{H}_3, H_m \rangle = \delta_{m,3}.$$

(4) Analogously, for $m' \geq 4$, we construct $\tilde{H}_{m'} \in \mathbb{H}^{p'}(D)$ such that

$$\langle \tilde{H}_{m'}, H_m \rangle = \delta_{m,m'}.$$

For $1 < p < \infty$, the fact that $\{H_m\}_{m\geq 0}$ is a basis in $\mathbb{H}^p(D)$ implies that, for $m' \geq 0$, $\tilde{H}_{m'} \in \mathbb{H}^{p'}(D)$ defines a linear continuous functional on $\mathbb{H}^p(D)$. By construction and Proposition 2.2, we have

Proposition 6.2 *For* $1 < p < \infty$ *and* $f \in \mathbb{H}^p(D)$,

$$f = \sum_{m\geq 0} \langle f, \tilde{H}_m \rangle H_m.$$

Then we can form maximal partial sum re-ordering algorithm, being a variation of AFD algorithm as follows. Our greedy AFD here is:

(1) Take f_0 and f_1 as the $\langle f, H_0\rangle$ and $\langle f, H_1\rangle = \langle f, \tilde{H}_{0,0}\rangle$. $x_0^f = 0$ and $x_1^f = 0$.
(2) For $m = 2$, we can write $m = 2^{j-1} + 1 + k$ where $j = 1, k = 0 < 2^{j-1}$. There exists only one k satisfying that $0 \le k < 2^{j-1}$ for $j = 1$. We take $f_2 = \langle f, \tilde{H}_2\rangle = \langle f, \tilde{H}_{1,0}\rangle$ and $x_2^f = 0$.
(3) For $3 \le m \le 4$, for $j = 2$, we have $2^{j-1} + 1 \le m \le 2^j$. We can write $m = 2^{j-1} + 1 + k$ where $j \ge 2, 0 \le k < 2^{j-1}$. There are two possibilities for k, we adopt greedy algorithm for $j = 2$. That is to say, we choose f_3 to be one of the quantities $\langle f, \tilde{H}_3\rangle = \langle f, \tilde{H}_{2,0}\rangle$ and $\langle f, \tilde{H}_4\rangle) = \langle f, \tilde{H}_{2,1}\rangle$ such that $|f_3| = \max(|\langle f, \tilde{H}_3\rangle|, |\langle f, \tilde{H}_4\rangle|)$. Then there exists $k_0 = 0$ or $k_0 = 1$ such that $\tilde{H}^f_{2,0} = \tilde{H}_{2,k_0}$ We denote $f_3 = \langle f, \tilde{H}^f_3\rangle = \langle f, \tilde{H}^f_{2,0}\rangle$ and $x_3^f = k_0$. We take $f_4 = \langle f, \tilde{H}^f_{2,1}\rangle$ to be the rest of the quantities $\langle f, \tilde{H}_3\rangle$ and $\langle f, \tilde{H}_4\rangle$ where $\tilde{H}^f_{2,1}$ is one of $\tilde{H}_{2,0}$ and $\tilde{H}_{2,1}$ such that $\tilde{H}^f_{2,1}$ is different to $\tilde{H}^f_3$. We take $x_4^f = \{0, 1\}\backslash k_0$.
(4) For $m \ge 5$, there exists $j \ge 3$ such that $2^{j-1} + 1 \le m \le 2^j$. We can write $m = 2^{j-1} + 1 + k$ where $j \ge 2, 0 \le k < 2^{j-1}$. For each $j \ge 3$, there exist at least 2^{j-1} possible of k satisfying that $0 \le k < 2^{j-1}$. For each j fixed, we adopt greedy algorithm for $0 \le k < 2^{j-1}$.
(5) That is to say, there exists $0 \le k_0 < 2^{j-1}$ such that

$$|\langle f, \tilde{H}_{j,k_0}\rangle| = \sup_{0\le k<2^{j-1}} |\langle f, \tilde{H}_{j,k}\rangle|.$$

We take $f_{2^{j-1}+1} = f_{j,0} = \langle f, \tilde{H}_{j,k_0}\rangle$, $\tilde{H}^f_{j,0} = \tilde{H}_{j,k_0}$, $H^f_{j,0} = H_{j,k_0}$ and $x^f_{2^{j-1}+1} = k_0$.
(6) Then there exists $0 \le k_1 < 2^{j-1}$ and $k_1 \ne k_0$ such that

$$|\langle f, \tilde{H}_{j,k_0}\rangle| = \sup_{0\le k<2^{j-1}, k\ne k_0} |\langle f, \tilde{H}_{j,k}\rangle|.$$

We take $f_{2^{j-1}+2} = f_{j,1} = \langle f, \tilde{H}_{j,k_1}\rangle$, $\tilde{H}^f_{j,1} = \tilde{H}_{j,k_1}$, $H^f_{j,1} = H_{j,k_1}$ and $x^f_{2^{j-1}+2} = k_1$.
(7) In turn, we find $f_m = f_{j,m-1-2^{j-1}}$ such that $\tilde{H}^f_{j,m-1-2^{j-1}} = \tilde{H}_{j,k_{m-1-2^{j-1}}}$ and $H^f_{j,m-1-2^{j-1}} = H_{j,k_{m-1-2^{j-1}}}$. We denote $x^f_m = k_{m-1-2^{j-1}}$.

Remark 6.3 (i) Note that the above optimal term ordering arrangement does not require unconditional basis property proved in Theorem 1.2. By invoking 1.2, however, we can also take the global maximum choice, but label x^f_m with both j^f and k^f. Such global approach is indeed faster for sparse data (wide sense), but a data sparsity condition must be added to analyze quantitatively.
(ii) Since our algorithm yields an unconditional basis (see Theorem 1.2), we can also approximate the function with a **maximum** $\mathbb{H}^p$ **norm**. In fact, assuming the first m functions $\{H^f_{m'}\}_{0\le m'\le m-1}$ are selected and we get $F_m = \sum_{0\le m'\le m-1}\langle f, \tilde{H}^f_{m'}\rangle H^f_{m'}$, we can always find another H^f_m such that the following

quantity is biggest possible

$$|f_0| + |f_1| + \| \left(\sum_{2 \le m' \le m} 2^j |\langle f, \tilde{H}^f_{m'} \rangle|^2 \chi(2^j x - x^f_{m'}) \right)^{\frac{1}{2}} \|_{L^p},$$

unless the decomposition of f has only a finite sum less than m term.

6.2 Proof of Main Theorem 1.2

We first prove (i). Let $\{H_m\}_{m \ge 0}$ be the rational functions defined in the Sect. 4.1.1. According to Proposition 4.8, the coefficients are combined according to Lusin's law of area integration, and the analytic space formed by $\{H_m\}_{m \ge 0}$ is contained in Hardy space $\mathbb{H}^p(D)$. According to Proposition 5.2, all functions in $\mathbb{H}^p(D)$ can be represented to be the linear combination of $\{H_m\}_{m \ge 0}$. Hence $\{H_m\}_{m \ge 0}$ is an unconditional basis of $\mathbb{H}^p(D)$.

We next prove (ii). Let $\{\tilde{H}_m\}_{m \ge 0}$ be the dual functions defined in the Sect. 4.1.1. Since $\{H_m\}_{m \ge 0}$ is an unconditional basis of $\mathbb{H}^p(D)$, the new algorithm through rearranging term order defined in Sect. 6.1 satisfies all the conditions in (ii) of Main Theorem 1.2.

Acknowledgements This project is partially supported by research grant of Macau University of Science and Technology (FRG-22-075-MCMS), Macao Government Research Funding (FDCT0128/2022/A), Science and Technology Development Fund of Macau SAR(005/2022/ALC), Science and Technology Development Fund of Macau SAR(0045/2021/A), Macau University of Science and Technology (FRG-20-021-MISE), National Key Research and Development Program of China (No. 2020YFA0714200) and Open Project Foundation of Intelligent Information Processing Key Laboratory of Shanxi Province (No. CICIP2018002).

Author Contributions BZ: initialized the question. HY, CH, Leong first discussed and formed the approach. TQ followed the discussion and completed the manuscript. All authors reviewed the manuscript.

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